# I. Optimal Rarity Meter Problem

Consider a single NFT collection. Let the collection consists of $N$ tokens. Some collections allow minting new tokens. We consider them at a fixed moment and design the rarity meter for a given blockchain snapshot. So the number of tokens $N$ is the number of tokens minted in the given snapshot, and it is fixed. Let

$$\mathbb{X}_N = \{X_1, \ldots, X_N\}$$

be tokens in the collection. Let each token $X_n$ has $T$ traits:

$$X_n = (x_{n1}, \ldots, x_{nT}).$$

The values of the traits are categorical variables–have no order. The traits are "gens" of the NFT, and their combination defines the "phenotype" of the token, for example, object shape or background color.

We say that an arbitrary function

$$R : \mathbb{X}_N \to [0, \infty)$$

is a **rarity meter**. One expects the bigger the value of $R(X_n)$ the rarer is the NFT $X_n \in \mathbb{X}_N$.

Different people may have different views on the rarity. So how can one compare two rarity meters and find the one which better meets expectations–the optimal rarity meter? The only answer we have is "the market knows." Users trade NFTs, and one expects to see a greater rarity meter for a more expensive NFT. Let $D$ deals be in a blockchain snapshot. Each deal $d = 1, \ldots, D$ characterizes with the deal moment $t_d$, the sold NFT index $i_d$ of and the deal price $p_d$. The price is expressed in cryptocurrency, and it is volatile. The interest in the collection changes over time as well. Thus, the NFTs' prices change with time too. We want such a final rarity meter that if two deals for different NFTs are close in time, the rarity meter value of the more expensive NFT is greater. The consideration of deals pairs $(d_1, d_2)$ with weights $k(t_{d_1}, t_{d_2})$ is a possible formalization, where weights decrease with time between deals $|t_{d_1} - t_{d_2}|$ increase. Let a function

$$\varphi \colon [0, \infty) \times [0, \infty) \to \mathbb{R}$$

compare rarities of two NFTs, where

$$\varphi(R_1, R_2) \to +\infty \text{ with } R_1 \to \infty,$$
$$\varphi(R_1, R_2) \to +\infty \text{ with } R_2 \to 0,$$

and $\varphi$ is a skew-symmetric function:

$$\varphi(R_2, R_1) = -\varphi(R_1, R_2).$$

Let a function

$$\psi \colon (0, \infty) \times (0, \infty) \to \mathbb{R}$$

compare prices of the deals of two NFTs, where

$$\psi(p_1, p_2) \to +\infty \text{ with } p_1 \to \infty,$$
$$\psi(p_1, p_2) \to +\infty \text{ with } p_2 \to 0,$$

and $\psi$ is a skew-symmetric function:

$$\psi(p_2, p_1) = -\psi(p_1, p_2).$$

Let $j = 1, \ldots, J$ be the index for all deal pairs $(d_{1_j}, d_{2_j})$, where

$$J = \frac{D(D-1)}{2},$$
$$d_{1_j}, d_{2_j} \le D,$$
$$d_{1_j} < d_{2_j}.$$

Denote $\vec{\varphi}, \vec{\psi}, \vec{k} \in \mathbb{R}^{J \times 1}$ vectors of all relative rarities, prices and weights, i.e.

$$\vec{\varphi} = \vec{\varphi}(R) = \Big(\varphi(R(X_{d_{1_1}}), R(X_{d_{2_1}})), \ldots, \varphi(R(X_{d_{1_J}}), R(X_{d_{2_J}}))\Big)^T$$
$$\vec{\psi} = \Big(\psi(p_{d_{1_1}}, p_{d_{2_1}}), \ldots, \psi(p_{d_{1_J}}, p_{d_{2_J}})\Big)^T$$
$$\vec{k} = \Big(k(t_{d_{1_1}}, t_{d_{2_1}}), \ldots, k(t_{d_{1_J}}, t_{d_{2_J}})\Big)^T.$$

Let $F(\vec{\varphi}, \vec{\psi}; \vec{k})$ be a similarity function of vectors $\vec{\varphi}$ and $\vec{\psi}$ with given weights $\vec{k}$. We propose the **optimal rarity meter problem** as follows

$$F(\vec{\varphi}(R), \vec{\psi}; \vec{k}) \to \max_{R \in \mathcal{R}}, \tag{1}$$

where $\mathcal{R}$ is a search space of rarity meters.

Denote $\vec{\mathbf{1}}_J \in \mathbb{R}$ the $J$-dimensional vector with all ones. Let for an arbitrary $\vec{a} \in \mathbb{R}^{J \times 1}$, a vector $\bar{\vec{a}}_{\vec{k}}$ be the mean vector $\vec{a}$ with weights $\vec{k}$

$$\bar{\vec{a}}_{\vec{k}} = \frac{\vec{k}^T \vec{a}}{\vec{k}^T \vec{\mathbf{1}}_J} \cdot \vec{\mathbf{1}}_J$$

and a vector $\dot{\vec{a}}_{\vec{k}}$ be the centered vector $\vec{a}$ with weights $\vec{k}$

$$\dot{\vec{a}}_{\vec{k}} = \vec{a} - \bar{\vec{a}}_{\vec{k}}.$$

Hereafter, we use

- Epanechnikov kernel as a weight function
$$k(t_1, t_2) = \begin{cases} \frac{3}{4}(1 - \frac{|t_1 - t_2|}{h})^2, & |t_1 - t_2| < h \\ 0, & |t_1 - t_2| \ge h \end{cases},$$
where $h = 7$ days,
- $\varphi(R_1, R_2) = \ln\left(\frac{1+R_1}{1+R_2}\right)$ as a relative rarity,
- $\psi(p_1, p_2) = \ln\left(\frac{p_1}{p_2}\right)$ as a relative price,
- $F(\vec{\varphi}, \vec{\psi}; \vec{k}) = \texttt{corr}(\vec{\varphi}, \vec{\psi}; \vec{k})$ as a weighted similarity of vectors, where $\texttt{corr}$ is the weighted correlation function. I.e., let $\mathbf{K} = \vec{k}\vec{\mathbf{1}}_K^T \in \mathbb{R}^{K \times K}$ be a diagonal matrix with elements from $\vec{k}$, then
$$F(\vec{\varphi}, \vec{\psi}; \vec{k}) = \frac{\dot{\vec{\varphi}}_{\vec{k}}^T \cdot \mathbf{K} \cdot \dot{\vec{\psi}}_{\vec{k}}}{\sqrt{\dot{\vec{\varphi}}_{\vec{k}}^T \cdot \mathbf{K} \cdot \dot{\vec{\varphi}}_{\vec{k}}} \cdot \sqrt{\dot{\vec{\psi}}_{\vec{k}}^T \cdot \mathbf{K} \cdot \dot{\vec{\psi}}_{\vec{k}}}}. \tag{2}$$

People already have their expectations of rarity. One can formulate preferences in partially ordered sets (posets), where the greater element is rarer than, the smaller one. For example, $X_i > X_j$ if $x_{id}$ has less entries in $\{x_{1d}, \ldots, x_{Nd}\}$ than $x_{jd}$. We equip each poset with a **score function**

$$s : \mathbb{X}_N \to \mathbb{R},$$

where

$$(X_i > X_j) \Rightarrow (s(X_i) > s(X_j)).$$

For example,

$$s_f(X_i) = \frac{N}{\#\{k|x_{kd} = x_{id}, k = 1, \ldots, N\}}, \tag{3}$$

where $\#A$ is the cardinality of the set $A$ and the subscript $f$ stands for frequency. A score function is a rarity meter but also expert-based and interpretable. We want an optimal interpretable rarity meter, so we take score functions as building blocks and define $\mathcal{R}$ as all the combinations of the chosen score functions with non-negative weights, i.e., a positive hull. Given $M$ score functions $s_1, \ldots, s_M$, the rarity meter **search space** equals

$$\mathcal{R} = \{R|R = \sum_{m=1}^{M} \alpha_m s_m, \text{ where } \alpha_m \in [0, \infty)\}. \tag{4}$$

## II. Tournament Score Function

Score functions define a rarity meter search space (4). A score function is the rarity meter, i.e., a mapping from NFT collection $\mathbb{X}_N$ to non-negative real numbers $[0, \infty)$. The expert-based and interpretability requirements are subjective. This section introduces the heuristic approach to the score functions.

### A. Unique vs Rare

Thinking about different collections, we noticed several things. Imagine a collection of 1 red ball and 999 gray balls. We think everyone will agree that this red ball is unique and very rare. But imagine a collection with 500 gray balls and 500 balls of different and non-repetitive colors. Look–any of these balls are unique–but to be honest, it is not what we would like. We feel these non-grays like "just 500 colored balls."

So we must not only think about ball color like a member of a group (like "there are only 20 such colors, this color is the same in the group of 20 same colored balls"), but also we must think about what groups are in our collection. For example 500 groups of 1 color and 1 group of 500 gray color.

Rarity.Tools formula suggests to count only an amount in the group: you get $\frac{\texttt{TOTAL NFTs}}{\texttt{group size}}$ point for every trait and sum of the points is your Rarity result. The sense of the very similar formula could be: "Ok, this color is a part of the group of $n$ same colors." Let us count the different colored balls and divide the group members' results. Formula is $\frac{\texttt{TOTAL NFTs - n}}{\texttt{n}}$ (and this is almost the same with the Rarity tools formula, difference is $-1$).

But this formula seems not good enough. Imagine an NFT with 2 traits. The first is color, and the second is shape. There are 5000 unique different colored and 5000 gray NFTs. 5000 "just colored" NFTs are all round, 4999 gray are triangles, and the last one-gray ball–is squared. If we count with the Rarity.tools formula, every "just colored" ball will have the same rarity score as a squared (and truly unique) gray ball.

If there are 500 groups of 20 same colored balls, we will not treat this as a big achievement. So it is important to differentiate between Uniqueness, Rarity, and Uniqueness Rarity. Uniqueness has no big value if it is not Rare. So we need the Rarity meter, which counts the rareness of this exact "color" and a collection structure.

### B. Score Function Design

Let us construct an alternative score function to the reverse frequency from Section II-A. Let the values of a trait be not comparable by themselves, i.e., and golden eyes are not better than brown eyes just because of color. We will compare only groups of the trait. If there are 50 golden eyes and 20 brown eyes, and 20 gray eyes, we will compare only 3 different groups of 50, 20, and 20 of this trait.

So, first of all, we split every NFT by traits and counted the score for each trait. After that, we take an average for all traits.

For every trait we write out all group sizes $N_1, \ldots, N_G$, where $G$ is the number of groups and $\sum_{g=1}^{G} N_g = N$. For example, for the collection with 1 golden, 20 blue, 21 green, 43 yellow, 1 gray and 14 white we write

$$(N_1, N_2, N_3, N_4, N_5, N_6) = (1, 20, 21, 43, 1, 14).$$

The order does not matter. Then, we start a tournament between these groups. One point is played out in every battle between groups with $N_1$ and $N_2$ elements correspondingly, and each group's points are inverse to the group size.

- $N_1$ wins $N_2/(N_1 + N_2)$ points
- $N_2$ wins $N_1/(N_1 + N_2)$ points.

For Example, 14 vs 21: 14 is more rare and gets $21/35$, 21 gets $14/35$. The full standings are in Table I. Note that the average of the right column is always 0.5.

Good news

- If we have 1 golden, 2 red, and 9997 gray, the difference in scores of 1 and 2 will not be too huge. For example, we see that 20 and 21 are very close.
- We can count the average rareness of an NFT with several traits.
- We are counting the percentage, not the score, depending on the collection size. The percentage is much clear, and you do not need to know additional information.

Bad news

- Sometimes results seem questionable. For example, Tables II, III, IV.

So if we leave the results as is, we will get a worse score for 5 Uniques (with 95 gray) vs. a group of 10 same-colored in $(10, 10, 20, 20, 20, 20)$ collection.

The finishing touch. As we mentioned before, the average of the right column is always 0.5. But the average of all entities of the trait (`trait_average`) is different. So if one trait is not more important than another, we need to weigh all traits. The final average of each trait should be the same–1. For every single instance we get `group_score/trait_average`. Formally, let $N(X)$ be the group size of $X$ for a given trait. The tournament score

$$s_t(X) = N \cdot \frac{-\frac{1}{2} + \sum_{g=1}^{G} \frac{N_g}{N(X)+N_g}}{\sum_{i=1}^{G} N_i \cdot \left(-\frac{1}{2} + \sum_{g=1}^{G} \frac{N_g}{N_i+N_g}\right)}. \tag{5}$$

TABLE I
STANDINGS FOR GROUPS WITH $1, 20, 21, 43, 1, 14$ ELEMENTS CORRESPONDINGLY

| $N_i/N_j$ | 1 | 20 | 21 | 43 | 1 | 14 | average |
|---|---|---|---|---|---|---|---|
| 1 | | 20/21 | 21/22 | 43/44 | 1/2 | 14/15 | 0.86 |
| 20 | 1/21 | | 21/41 | 43/63 | 1/21 | 14/34 | 0.34 |
| 21 | 1/22 | 20/41 | | 43/64 | 1/22 | 14/35 | 0.33 |
| 43 | 1/44 | 20/63 | 21/64 | | 1/44 | 14/57 | 0.19 |
| 1 | 1/2 | 20/21 | 21/22 | 43/44 | | 14/15 | 0.86 |
| 14 | 1/15 | 20/34 | 21/35 | 43/57 | 1/15 | | 0.42 |

TABLE II
STANDINGS FOR 50 GREY AND 50 "JUST COLORED" ELEMENTS

| | | 50 groups | | |
|---|---|---|---|---|
| | $N_i/N_j$ | 50 | 1 | average |
| | 50 | | 0.02 | 0.02 |
| 50 groups | 1 | 0.98 | 0.5 | 0.51 |

The term $-\frac{1}{2}$ is to eliminate the tournament within the group. See Tables V, VI, VII.

Different entities of the trait have relatively the same score, but also the average of different traits is the same and equal to 1. Also, note that More Rare Uniqueness will have a better score than Less Rare Uniqueness.

TABLE III
STANDINGS FOR 4 GROUPS OF 20 AND 2 GROUPS OF 10

| | 20 | 20 | 20 | 20 | 10 | 10 | average |
|---|---|---|---|---|---|---|---|
| 20 | | 0.50 | 0.50 | 0.50 | 0.33 | 0.33 | 0.43 |
| 20 | 0.50 | | 0.50 | 0.50 | 0.33 | 0.33 | 0.43 |
| 20 | 0.50 | 0.50 | | 0.50 | 0.33 | 0.33 | 0.43 |
| 20 | 0.50 | 0.50 | 0.50 | | 0.33 | 0.33 | 0.43 |
| 10 | 0.67 | 0.67 | 0.67 | 0.67 | | 0.50 | 0.63 |
| 10 | 0.67 | 0.67 | 0.67 | 0.67 | 0.50 | | 0.63 |

TABLE IV
STANDINGS FOR 5 GROUPS OF 1 AND 1 GROUP OF 95

| | 1 | 1 | 1 | 1 | 1 | 95 | average |
|---|---|---|---|---|---|---|---|
| 1 | | 0.50 | 0.50 | 0.50 | 0.50 | 0.99 | 0.598 |
| 1 | 0.50 | | 0.50 | 0.50 | 0.50 | 0.99 | 0.598 |
| 1 | 0.50 | 0.50 | | 0.50 | 0.50 | 0.99 | 0.598 |
| 1 | 0.50 | 0.50 | 0.50 | | 0.50 | 0.99 | 0.598 |
| 1 | 0.50 | 0.50 | 0.50 | 0.50 | | 0.99 | 0.598 |
| 95 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | | 0.010 |

TABLE V
AVERAGE STANDINGS FOR 50 GRAY AND 50 "JUST COLORED" ELEMENTS

| | | group score | normalized score |
|---|---|---|---|
| | 50 | 0.0196 | 0.0741 |
| 50 times | 1 | 0.5096 | 1.9259 |
| | trait average | 0.2646 | |

TABLE VI
AVERAGE STANDINGS FOR 4 GROUPS OF 20 AND 2 GROUPS OF 10

| | group score | final rarity |
|---|---|---|
| 20 | 0.4333 | 0.9154 |
| 20 | 0.4333 | 0.9154 |
| 20 | 0.4333 | 0.9154 |
| 20 | 0.4333 | 0.9154 |
| 10 | 0.6333 | 1.3380 |
| 10 | 0.6333 | 1.3380 |
| trait average | 0.4733 | |

| | group score | final rarity |
|---|---|---|
| 1 | 0.5979 | 15.0262 |
| 1 | 0.5979 | 15.0262 |
| 1 | 0.5979 | 15.0262 |
| 1 | 0.5979 | 15.0262 |
| 1 | 0.5979 | 15.0262 |
| 95 | 0.0104 | 0.0262 |
| trait average | 0.0398 | |

TABLE VII
AVERAGE STANDINGS FOR 5 GROUPS OF 1 AND 1 GROUP OF 95